



GENERALIZED DERIVATIONS ACTING AS HOMOMORPHISM AND ANTI-HOMOMORPHISM ON LIE IDEALS OF PRIME RINGS

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Abstract

The major goal of this paper is to demonstrate the following conclusion:

Allow U to be a non-zero square closed Lie ideal of a 2-torsion free prime ring R , and f to be a generalized (U, R) -derivation of R . If f acts as a homomorphism and as an anti-homomorphism on U , then either $f = 0$ or $U \subset Z(R)$.

Keywords: Generalized (U, R) -Derivation, Prime Ring, Lie Ideal, Square Closed Lie Ideal, Homomorphism, Anti-Homomorphism.

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Introduction

In the year 1950s I. N. Herstein [9] pioneered research into the relationship between the Jordan and Lie structure of associative rings. Herstein [10] showed a classic finding in this direction in 1957, which served as a jumping off point for many other researchers.

The relationship between usual derivations and Lie ideals of prime rings has been extensively studied in the last 30 years, in particular,

when this relationship involves the action of the derivation on Lie ideals.

R. Awtar[3] extended the Herstein's theorem to Lie ideals. He proved that if U is a Lie ideal of a prime ring R of characteristic changed from 2 such that $u^2 \in U$, for each $u \in U$ and $d : R \rightarrow R$ is an additive mapping such that d is a Jordan derivation on U then d is a derivation on U .

The notion of generalized derivations on a ring R which was introduced by Bresar [6] is connected to a derivation of R . In [13], without applying derivations, Nakajima established another form of generalized derivations and offered categorical features to those generalized derivations. When R has an identity element, these two concepts are congruent. Nakajima's [13] results were extended to generalized Jordan and Lie derivations..L. Oukhtite, S. Salhi and L. Taofiq[14] studied Jordan generalized derivations on sigma- prime rings and proved that every Jordan generalized derivation on a Lie ideal U of sigma- prime ring R is a generalized derivation on U of R .

Lately, Ashraf and Rehman [1] looked into the Herstein[10] problem for a Jordan generalized derivation. They proved that any Jordan generalized derivation on R is a generalized derivation in a 2-torsion-free ring R with a nonzero commentator divisor.

Well along, Cortes and Haetinger [7] refined Ashraf's theorem to higher derivations in general. They established that every Jordan generalized higher derivation on R is a generalized higher derivation if R is a 2-torsion-free ring with a commentator right nonzero divisor.

(U, R) -derivations in rings have been presented by Faraz, Haetinger and Majeed[8] as a simplification of Jordan derivations on a Lie ideal of a ring. Faraj, Haetinger and Majeed [8] showed that if R is a prime ring, $\text{char}(R) \neq 2$, U a square closed Lie ideal of R and d is a (U, R) -derivation of R , then $d(ur) = d(u)r + ud(r)$, $\forall, u \in U, r \in R$. This result is a generalization of a result of Awtar's [3] theorem. M. M. Rahman and A. C. Paul[16] have introduced (U, M) derivation and general (U, M) derivation in Γ ring and generalized the main theorems of Faraz, Haetinger and Majeed[8].

Preliminaries

Let us study R to be an associative ring with center $Z(R)$ through the project. A ring R is believed to be 2-torsion free if $2x = 0$ with $x \in R$,

formerly $x=0$. A ring R is called a prime ring if for any $x, y \in R$, $xRy=0$ implies $x=0$ or $y=0$. In a ring R , the symbol $[x, y]$ is known as the commutator of x and y , which is defined by $[x, y]=xy-yx$, where $x, y \in R$. Two useful basic commutator identities are:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z.$$

An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal U of R is called a square closed Lie ideal if $u^2 \in U$ for all $u \in U$. Furthermore, if the Lie ideal U of R is square closed and U is not contained in $Z(R)$, where $Z(R)$ denotes the center of R , then U is called an admissible Lie ideal of R .

An additive mapping $d: R \rightarrow R$ is said to be a derivation if $d(xy) = d(x)y + yd(x)$ for all $x, y \in R$. An additive mapping $f: R \rightarrow R$ is called a generalized derivation if there exists a derivation $d: R \rightarrow R$ such that $f(xy) = f(x)y + yd(x)$ holds for all $x, y \in R$.

An additive mapping $d: R \rightarrow R$ is said to be a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x \in R$ and it is called a Jordan triple derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

An additive mapping $d: R \rightarrow R$ is said to be a (U, R) -derivation of R if $d(ur + su) = d(u)r + ud(r) + d(s)u + sd(u)$, $\forall u \in U$ & $\forall r, s \in R$, U is a Lie ideal of R .

An additive mapping $f: R \rightarrow R$ is said to be a generalized (U, R) -derivation of R if there exists a (U, R) -derivation d of R such that $f(ur + su) = f(u)r + ud(r) + f(s)u + sd(u)$, for all $u \in U$, $\forall r, s \in R$.

Let S be a non-empty subset of R and f is a generalized derivation of R . If $f(xy) = f(x)f(y)$ [resp. $f(xy) = f(y)f(x)$] for all $x, y \in S$, then f is said to act as a homomorphism [resp. as an anti-homomorphism] on S .

A mapping $\sigma: R \rightarrow R$ is called an involution if $\sigma(a+b) = \sigma(a) + \sigma(b)$, $\sigma^2(a) = a$, $\sigma(ab) = \sigma(b)\sigma(a)$ holds for all $a, b \in R$. A Lie ideal U of R is called a σ -Lie ideal if $\sigma(U) = U$ and it is called a σ -square closed Lie ideal if it is a σ -Lie ideal and for all

$u \in U$ implies $u^2 \in U$. A ring R with involution σ is said to be a σ -prime ring if $aRb = aR\sigma(b) = \{0\}$ implies that $a = 0$ or $b = 0$. It is worthwhile to note that every prime ring having an involution σ is σ -prime but the converse is not true in general.

Bresar [6] developed the concept of generalized derivation, and B. Hvala [11] and T. K. Lee [12] derived numerous characterizations of generalized derivation. Bell and Kappe show in [4] that if a derivation is both a homomorphism and an anti-homomorphism on a prime ring R 's non-zero ideal I , then $d=0$. This conclusion is extended to a square closed Lie ideal by Asma, Rehman, and Shakir [2], while Rehman [17] establishes the same result for generalized derivations. Using similar ideas, Paul and Chakraborty [15] expand the main result to the square closed Lie ideal.

In this paper, we spread the key result of previous workers [2, 4, 15, 17] to square closed Lie ideal with (U, R) -derivation of R by using the similar arguments.

(U, R) Derivation

By demonstrating that if R is a prime ring, $\text{char}(R) \neq 2$, U is a square closed ring, Faraj, C. Haetinger, and H. Majed [8] familiarize and learn the notion of (U, R) -derivation. If R is the Lie ideal and d is a (U, R) -derivation of R , then $d(ur) = d(u)r + ud(r)$, for all $u \in U, r \in R$, extending Awtar's result ([3, Theorem]).

Definition: Suppose U be a Lie ideal of a ring vR . An additive mapping $d : R \rightarrow R$ is thought to be a (U, R) -derivation ((U, R) - D , for short) of R if $d(ur + su) = d(u)r + ud(r) + d(s)u + sd(u)$, for all $u \in U, \forall r, s \in R$.

Example. Let R be a ring of all 2×2 matrices over a commutative ring S . Let

$$U = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in S \right\}$$

Then U is a Lie ideal of R . Define $d : R \rightarrow R$ by $d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$

Then d is a (U, R) derivation.

$$\text{Let } u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in U \quad r = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in R \quad s = \begin{pmatrix} p & q \\ r & t \end{pmatrix} \in R$$

$$\begin{aligned} \text{Then } d(ur + su) &= d \begin{pmatrix} ax + bz + pa + qc & ay + bw + pb - qa \\ cx - az + ra + tc & rb - ta \end{pmatrix} \\ &= \begin{pmatrix} 0 & -ay - bw - pb + qa \\ cx - az + ra + tc & 0 \end{pmatrix} \end{aligned}$$

$$\text{and } d(u)r + ud(r) + d(s)u + sd(u) =$$

$$\begin{aligned} &\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} + \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix} + \begin{pmatrix} 0 & -q \\ z & 0 \end{pmatrix} + \begin{pmatrix} 0 & -q \\ r & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} p & q \\ r & t \end{pmatrix} \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \\ &= \begin{pmatrix} -bz + bz - qc + qc & -bw - ay + qa - pb \\ cx - az + ra + tc & cy - cy + rb - rb \end{pmatrix} \\ &= \begin{pmatrix} 0 & -ay - bw + qa - pb \\ cx - az + ra + tc & 0 \end{pmatrix} \end{aligned}$$

Thus $d(ur + su) = d(u)r + ud(r) + d(s)u + sd(u)$, for all $u \in U, \forall r, s \in R$.

So d is a (U, R) -derivation.

We need the following Lemmas due to Faraj, C. Haetinger and H. Majed [8] for proving our results.

Lemma 3.1: Faraj, C. Haetinger and H. Majed [8]. Assume R is a 2-torsion-free ring with d being (U, R) - D of R . Then $d(uru) = d(u)ru + ud(r)u + urd(u)$, for all $u \in U, r \in R$.

Lemma 3.2: Faraj, C. Haetinger and H. Majed [8]. Assume R is a prime ring, $\text{char}(R) \neq 2$, and U is a closed square, R 's Lie Ideal and d be (U, R) - D of R . Formerly $(u^2, r) = 0$, for all $u \in U, r \in R$.

Lemma 3.3: Faraj, C. Haetinger and H. Majed [8]. Suppose that R be a prime ring, $\text{char}(R) \neq 2$ be a nonzero allowable Lie ideal of R , and U be a nonzero admissible Lie ideal of R . Then R 's nonzero ideal is contained in U .

Lemma 3.4: Faraj, C. Haetinger and H. Majed [8]. Suppose that R be a prime ring, $\text{char}(R) \neq 2$, and $U \not\subseteq Z(R)$ exist Lie Ideal. At that time there be elements $a, b, c \in U$ such that $[a, b, c] = abc - cba \neq 0$.

Lemma 3.5: Faraj, C. Haetinger and H. Majed [8]. Assume R is a prime ring, $\text{char}(R) \neq 2$ be a nonzero allowable Lie ideal of R . At that time for several $t \in R$, if $tv^2 + v^2t = 0$, for all $v \in U, t = 0$

Proof: Linearize $tv^2 + v^2t = 0$ on v , then

$$t(uv^2 + v^2u) + (uv^2 + v^2u)t = 0$$

Since $tv^2 = -v^2t \forall v \in U$, so the above equation becomes

$$0 = tuv^2 - v^2tu - utv^2 + v^2tu$$

$$= (tu - ut)v^2 - v^2(tu - ut) = 0 \quad \text{or, } [t, u]v^2 - v^2[t, u] = 0$$

$$\text{or, } -v^2[t, u] - v^2[t, u] = 0 \quad \text{or, } \mathcal{L}v^2[t, u] = 0 \quad \text{or, } v^2[t, u] = 0 \quad \text{If we}$$

linearize $tv^2 = 0$ on v , then $0 = t(u + v)^2 = t(u^2 + uv + vu + v^2) = t(uv + vu) = (uv + vu)u = tuv + utv = tuv + utv = (tu)v(tu), \forall u, v \in U, \forall t \in R$. As U holds a nonzero R ideal, and R be a prime, so $tu = 0$.

Imprimness of R , we get $t = 0$.

Faraj, C. Haetinger and H. Majed extend ([3, Theorem] to (U, R) derivations in the following way:

Theorem 3.1: Assume R is a 2-torsion-free ring with d being $(U, R) - D$ of R . Then $d(uru) = d(u)ru + ud(r)u + urd(u)$, for all $u \in U, r \in R$.

Proof. Then d be a (U, R) derivation of R .

$$d(uur + uru) = d(u)ur + ud(ur) + d(ur)u + urd(u). \quad (6)$$

On the other hand,

$$\begin{aligned} d(uur + uru) &= d(u^2r) + d(ur)u \\ &= d(u^2r) + d(u)ru + ud(r)u + urd(u). \end{aligned} \quad (7)$$

Now by using Lemma 2.4, equation (7) becomes

$$\begin{aligned} d(u^2r + uru) &= d(u^2)r + u^2d(r) + d(u)ru + ud(r)u + urd(u) \\ &= d(u)ur + ud(u)r + u^2d(r) + d(u)ru + ud(r)u + urd(u). \end{aligned} \quad (8)$$

By comparing (6) and (8), we get

$$u\Phi(u, r) + \Phi(u, r)u = 0, \forall u \in U, r \in R. \quad (9)$$

By linearizing (9) on u , we have

$$u\Phi(v, r) + v\Phi(u, r) + \Phi(u, r)v + \Phi(v, r)u = 0. \quad (10)$$

Replace v by v^2 in (10), then and there by Lemma 2.4, we get $v^2\Phi(u, r) + \Phi(u, r)v^2 = 0$, for all $u, v \in U, r \in R$.

Currently, if $U \not\subset Z(R)$ then and there by using Lemma 2.7, we become $\Phi(u, r) = 0$, for all $u \in U, r \in R$.

Generalized (U, R) derivaion

By establishing that if R is a prime ring, $\text{char}(R) \neq 2$, U is a square closed Lie ideal of R , Faraj, C. Haetinger, and H. Majed [8] develop and analyze the concept of generalized (U, R) -derivation. then $f(ur) = f(u)r + ud(r)$, for all $u \in U, r \in R$.

Definition: Suppose that, U be a Lie ideal of a ring R and f be an additive mapping R into itself. Then f is said to be a generalized (U, R) -derivation ($G(U, R) - D$, for short) of R if there exists a (U, R) -derivation d of R such that $f(ur + su) = f(u)r + ud(r) + f(s)u + sd(u)$, for all $u \in U, \forall r, s \in R$.

Example. Assume R is a ring containing all 2×2 matrices over a commutative ring S . Assume $U = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in S \right\}$

Then U is R 's Lie ideal. Express $f : R \rightarrow R$ in $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}$ for

all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$. Formerly there is a (U, R) -derivation d of R which is

definite by $d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$.

Then f is a derivation of R that is generalized (U, R) .

Let $u = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in U$ $r = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in R$ $s = \begin{pmatrix} p & q \\ r & t \end{pmatrix} \in R$

$$\begin{aligned} \text{Then } f(ur + su) &= f \begin{pmatrix} ax + bz + pa + qb & ay + bw + pb + qa \\ bx + az + ra + tb & by + aw + br + ta \end{pmatrix} \\ &= \begin{pmatrix} ax + bz + pa + qb & 0 \\ 0 & -by - aw - br - ta \end{pmatrix} \end{aligned}$$

$$\text{and } f(u)r + ud(r) + f(s)u + sd(u) =$$

$$\begin{aligned} &\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & -t \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} + \begin{pmatrix} p & q \\ r & t \end{pmatrix} \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \\ &= \begin{pmatrix} ax + bz + pa + qb & ay - ay + pb - pb \\ -az + az - tb + tb & -by - aw - br - ta \end{pmatrix} \\ &= \begin{pmatrix} ax + bz + pa + qb & 0 \\ 0 & -by - aw - br - ta \end{pmatrix} \end{aligned}$$

Thus $f(ur + su) = f(u)r + ud(r) + f(s)u + sd(u)$, for all $u \in U, \forall r, s \in R$. So, f is a generalized (U, R) derivation of R .

Lemma 4.1 [5, Lemma 4]. Consider R to be a prime ring. $\text{char}(R) \neq 2$. Then U is a R 's Lie ideal, $U \not\subseteq Z(R)$. Uncertainty $a, b \in R$ then $aUb = 0$ then either $a = 0$ or $b = 0$.

Remark: If f is a generalized (U, R) derivation of R for each $u \in U, r \in R$ authors signify $\delta(u, r)$ the part of R definite in $\delta(u, r) = f(u)r + ud(r)$, for all $u \in U, r \in R$.

Lemma 4.2: Consider R to be a prime ring, $\text{char}(R) \neq 2$, U is a allowable Lie ideal of R and f be a general (U, R) derivation of R . Formerly $\delta(u, v) = 0$, for all $u, v \in U$.

Theorem 4.1. Consider R to be a prime ring, $\text{char}(R) \neq 2$, U is a square secure Lie ideal of R and f be a generalized (U, R) derivation of R . Formerly $f(ur) = f(u)r + ud(r)$, for all $u \in U, r \in R$.

Proof. Since above Lemma 4.2 then the previous Statement, we have

$$\delta(u, v) = 0, \quad \forall u, v \in U. \quad (1)$$

Replacing v by $ur - ru$ in (1), then we get

$$\begin{aligned}
0 &= \delta(u, ur - ru) \\
&= f(u^2 r) - f(ur) - f(u)(ur - ru) - ud(ur - ru).
\end{aligned}$$

Since d be a (U, R) derivation of R , so

$$f(u^2 r) = f(u)ur + ud(u)r + u^2 d(r) \quad (2)$$

Now, let $x = uur + uru$. Then in the characterization of general (U, R) derivation of R , $f(x) = f(u)ur + ud(ur) + f(ur)u + urd(u)$

$$= f(u)ur + ud(u)r + u^2 d(r) + f(ur)u + urd(u) \quad (3)$$

On the other hand, we have

$$\begin{aligned}
f(x) &= f(u^2 r) + f(ur) \\
&= f(u)ur + ud(u)r + u^2 d(r) + f(u)ru + ud(r)u + urd(u)
\end{aligned} \quad (4)$$

Comparing (3) and (4), we get

$$\delta(u, r)u = 0, \quad \forall u \in U, r \in R. \quad (5)$$

Linearizing (5) on u , we get

$$\delta(u, r)v + \delta(v, r)u = 0. \quad (6)$$

Replace v by v^2 in (6).

Since $\delta(u^2, r) = 0$, then

$$\delta(u, r)v^2 = 0, \quad \forall u, v \in U, r \in R. \quad (7)$$

Uncertainty U is a significant Lie ideal; formerly $\delta(u, r) = 0$, for all $u \in U, r \in R$. Uncertainty U is non central, then change v by $u + v$ now equality (7) to become $\delta(u, r)vu = 0$, for all $u, v \in U, r \in R$.

Since U benon-centrally, then and there $\delta(u, r) = 0$, for all $u \in U, r \in R$.

Main Results

We now apply the basic conclusion of [15] to the square closed Lie ideal, using a generalized (U, R) -derivation of R . The next two lemmas (already established) are required to achieve the chosen results.

Lemma 5.1 (J. Bergun, I. N. Herstein and J. W. Kerr[5], Lemma 4) Suppose that $U \not\subset Z(R)$ is a Lie ideal of a 2-torsion allowed prime ring R then $a, b \in R$ such that $aUb = 0$, at that point $a = 0$ or $b = 0$.

Lemma 5.2 (J. Bergun, I. N. Herstein and J. W. Kerr [5], Lemma 5) Assume $U \neq 0$ is a Lie ideal of a 2-torsion allowed prime ring R then $d \neq 0$ a derivation of R such that $d(U) = 0$. At that point $U \subset Z(R)$.

The next suitable outcome plays as significant part to influence the objective.

Lemma 5.3 Uncertainty $U \neq 0$ be a Lie ideal of a 2-torsion allowed prime ring R such that $[U, U] = 0$, at that point $U \subset Z(R)$.

Proof: For all $u \in U$ and $x \in R$, we require

$$[u, [u, x]] = 0 \quad (1)$$

Replacing x by xy with $y \in R$, we obtain

$$\begin{aligned} 0 &= [u, x[u, y] + [u, x]y] \\ &= x[u, [u, y]] + [u, x]u, y + [u, x]u, y + [u[u, x]]/y \\ &= 2[u, x]u, y \end{aligned}$$

Then R be 2-torsion free, we become

$$[u, x]u, y = 0 \quad (2)$$

Such that $u \in U$ also $x, y \in R$.

Placing yz used for y in (2) with $z \in R$, then by (2), we get

$$[u, x]y[u, z] = 0 \text{ for all } u \in U \text{ and } x, y, z \in R.$$

Thus, we have $[u, x]R[u, z] = 0$

So, $u, x = 0$ or $[u, z] = 0$ i.e $u \in U$ also $x, z \in R$ (in the primness of R).

Now equally the cases, we get that $U \subset Z(R)$.

We may now demonstrate our major finding in the following manner.

Theorem: Condition $U \neq 0$ is a closed square Lie ideal of a 2-torsion free prime ring R then f is a comprehensive (U, R) -derivation of R , at that point

- (i) f Actions as per a homomorphism proceeding $U \Rightarrow$ either $f = 0$ or $U \subset Z(R)$.
- (ii) f actions as per an anti-homomorphism proceeding $U \Rightarrow$ either $f = 0$ or $U \subset Z(R)$.

Proof: There exists a (U, R) -derivation d of R such that f is a generalized (U, R) -derivation of a 2-torsion free prime ring R , where U is a square closed Lie ideal of R .

$$f(ur) = f(u)r + ud(r), \text{ for all } u \in U, r \in R \quad (1)$$

$$\text{and } d(ur) = d(u)r + ud(r), \text{ for all } u \in U, r \in R \quad (2)$$

Replacing r and s by $(2u)r + r(2u)$ in (1), and denote w by $u((2u)r + r(2u)) + ((2u)r + r(2u))u$, then

$$\begin{aligned} f(w) &= f(u((2u)r + r(2u)) + ((2u)r + r(2u))u) \\ &= 2(f(u)(ur + ru) + ud(ur + ru)u + f(ur + ru)u + (ur + ru)d(u)) \\ &= 2(f(u)(ur + ru) + u(d(u)r + ud(r) + d(r)u + ud(u) + \\ &\quad 2((f(u)r + ud(r) + f(r)u + rd(u))u + (ur + ru)d(u))) \end{aligned}$$

On the other hand

$$\begin{aligned} f(w) &= f(u((2u)r + r(2u)) + ((2u)r + r(2u))u) \\ &= f((2u^2)r + r(2u^2)) + 4f(uru) \\ &= f(2u^2)r + (2u^2)d(r) + f(r)(2u^2) + rd(2u^2) + 4f(uru) \\ &= 2(f(u)ur + ud(u)r + u^2d(r) + f(r)u^2 + rud(u) + rd(u)u) + 4f(uru) \end{aligned}$$

Comparing above two relations and Using R 's 2 torsion freeness, we become

$$f(uru) = f(u)ru + ud(r)u + urd(u) \quad (3)$$

Now linearizing (3) with respect u , we get

$$f(urv + vru) = f(u)rv + ud(r)v + urd(v) + f(v)ru + vd(r)u + vrd(u) \quad (4)$$

Let $x = 4(uvrvu + vuwuv)$, then by (3)

$$f(x) = f((2uv)w(2vu) + (2vu)w(2uv))$$

$$\begin{aligned}
&= f(2uv)w(2vu) + (2uv)d(w)(2vu) + 2uvw d(2vu) + \\
&\quad f(2vu)w(2uv) + (2vu)d(w)(2uv) + 2vuw d(2uv)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f(x) &= f((2uv)w(2vu) + (2vu)w(2uv)) \\
&= f(u(4vwv)u + v(4uwu)v) \\
&= f(u)(4vwvu) + ud(4vwv)u + 4uvw d(u) + f(v)(4uwuv) + \\
&\quad vd(4uwu)v + 4vuw d(v).
\end{aligned}$$

Comparing the exact sides of $f(x)$ and then d is a (U, R) derivation of R , thus

$$\begin{aligned}
0 &= 4(\delta(u, v)wvu + \delta(v, u)wuv + uvw\Phi(v, u) + vuw\Phi(u, v)) \\
&= 4(\delta(u, v)w[u, v] + [u, v]w\Phi(u, v))
\end{aligned}$$

Then R be a 2-torsion-free and through Theorem 3.1, we need $\delta(u, v)w[u, v] = 0$. then U be a non-centrally Lie ideal, at that point by Lemma 4.1,

$$\delta(u, v) = 0, \text{ for all } u, v \in U.$$

Replacing v by $ur - ru$ in $\delta(u, v) = 0$, we get

$$\begin{aligned}
0 &= \delta(u, ur - ru) \\
&= f(u^2r) - f(ur) - f(u)(ur - ru) - ud(ur - ru).
\end{aligned}$$

Since d be a (U, R) derivation of R , so

$$f(u^2r) = f(u)ur + ud(u)r + u^2d(r)$$

Now, let $x = uur + uru$. Then by the definition of generalized (U, R) derivation of R , $f(x) = f(u)ur + ud(ur) + f(ur)u + urd(u)$

$$= f(u)ur + ud(u)r + u^2d(r) + f(ur)u + urd(u)$$

On the other hand, we have

$$\begin{aligned} f(x) &= f(u^2 r) + f(ur u) \\ &= f(u)ur + ud(u)r + u^2 d(r) + f(u)ru + ud(r)u + urd(u) \end{aligned}$$

Comparing right hand sides of $f(x)$, we get

$$\delta(u, r)u = 0, \quad \forall u \in U, r \in R$$

Linearizing with respect to u , we get $\delta(u, r)v + \delta(v, r)u = 0$

Replace v by v^2

Then $\delta(u^2, r) = 0$, at that point

$$\delta(u, r)v^2 = 0, \quad \forall u, v \in U, r \in R$$

Uncertainty U is an essential Lie ideal, then $\delta(u, r) = 0$, for all $u \in U, r \in R$. If U is non central, then change v by $u + v$ to get

$$\delta(u, r)vu = 0, \quad \text{for all } u, v \in U, r \in R.$$

Since U is non-central, then $\delta(u, r) = 0$, for all $u \in U, r \in R$ and consequently $f(ur) = f(u)r + ud(r)$, for all $u \in U, r \in R$.

If U is not contained in $Z(R)$ then $U \not\subset Z(R)$

Then U be a square closed Lie ideal, we need

$$uv + vu = (u + v)(u + v) - u^2 - v^2 \in U \text{ for all } u, v \in U.$$

Also, we get $uv - vu \in U$ for all $u, v \in U$.

So, $2uv \in U$ for all $u, v \in U$.

Thus, $4(uvw) = 2(2uv)w \in U$ for all $u, v, w \in U$.

If f actions as per a homomorphism on U , at that point we get

$$\begin{aligned} f(4uvw) &= f(2(2uv)w) = 4f(uv)w + 4uvd(w) \\ &= 4(f(u)f(v)w + uvd(w)), \quad \text{for all } u, v, w \in U \end{aligned} \tag{5}$$

On the other hand,

$$f(4uvw) = f(2u)2vw = 4f(u)f(vw)$$

$$= 4(f(u)f(v)w + f(u)vd(w)), \forall u, v, w \in U \quad (6)$$

When we multiply (5) and (6) by R 's 2-torsion freeness, we become

$$f(u)vd(w) = uv d(w),$$

Then it becomes

$$(f(u) - u)vd(w) = 0, \forall u, v, w \in U.$$

Thus, we have $(f(u) - u)Ud(w) = 0$ for all $u, w \in U$.

Now opinion of Lemma 5.1, we achieve that

$$f(u) - u = 0 \text{ for all } u \in U \text{ or } d(w) = 0 \text{ for all } w \in U$$

If $d(w) = 0$ for all $w \in U$, at that point by Lemma 5.2, we need $d = 0$ or $U \subset Z(R)$.

Since $U \not\subset Z(R)$, we get $d = 0$.

On the other hand, if $f(u) - u = 0$ for all $u \in U$, then we have

$$f(u) = u, \forall u \in U.$$

Replacing u by $2uv$ for $v \in U$ and using the 2-torsion freeness of R , we get

$$uv = f(uv) = f(u)v + ud(v) = uv + ud(v) \text{ for all } u, v \in U.$$

So, we need $ud(v) = 0$ for all $u, v \in U$. Consequently, $Ud(v) = 0$ for all $v \in U$.

Then $[U, R] \subset U$, we get $[U, R]d(v) = 0$ for all $v \in U$.

So, $URd(v) = 0$ for all $v \in U$.

As $U \neq 0$ and R is a prime ring, we have $d(v) = 0$ for all $v \in U$.

Thus, by Lemma 5.2, we get $U \subset Z(R)$.

Let us suppose the f acts as an anti-homomorphism on U . Then we have

$$f(u)v + ud(v) = f(v)f(u) = f(uv), \forall u, v \in U \quad (7)$$

We get (7) by substituting $2uv$ for u in (7) and using (7).

$$uvd(v) = f(v)ud(v), \forall u, v \in U \quad (8)$$

Substituting $2wu$ for u in (8), we get

$$wuvd(v) = f(v)wud(v), \forall u, v \in U$$

Multiplying (8) by w on the left, we have

$$wuvd(v) = wf(v)ud(v), \text{ for all } u, v \in U$$

Comparing above two relations, we get

$$[w, f(v)]ud(v) = 0, \forall u, v \in U$$

Now opinion of Lemma 5.1, we become

$$[w, f(v)] = 0 \text{ for all } v, w \in U \text{ or } d(v) = 0 \text{ for all } v \in U.$$

If $d(v) = 0$ for all $v \in U$, at that point by Lemma 5.2, we invention $d = 0$ or $U \subset Z(R)$, and therefore $d = 0$. Then $U \not\subset Z(R)$

On the other hand, if $[w, f(v)] = 0$

For all $u, v \in U$, then changing v by $2vw$, we need

$$v[w, d(w)] + [w, v]d(w) = 0, \forall u, v \in U$$

Again, substituting $2uv$ for $v \in U$, and by 2-torsion freeness of R , we become

$$\begin{aligned} 0 &= uv[w, d(w)] + [w, uv]d(w) \\ &= uv[w, d(w)] + u[w, v]d(w) + [w, u]vd(w) \\ &= u(v[w, d(w)] + [w, v]d(w)) + [w, u]vd(w) = [w, u]vd(w) \end{aligned}$$

Thus, we obtain $[w, u]Ud(w) = 0, \forall u, v \in U$

Applying Lemma 5.1, we find that

$$d(w) = 0, \forall w \in U \quad \text{or} \quad [w, u] = 0, u, w \in U$$

If $[w, u] = 0$ for all $u, w \in U$, In the opinion of Lemma 5.3, at that point, it follows that $U \subset Z(R)$, which is a conflict to the fact that $U \not\subset Z(R)$. Thus, we need $d(w) = 0, \forall w \in U$.

Inby Lemma 5.2, we have $d = 0$ or $U \subset Z(R)$.

Then $U \not\subset Z(R)$, we complete that $d = 0$.

As a result, if $U \neq 0$ is the square closed Lie ideal of a 2-torsion free prime ring R , and f is the generalized (U, R) -derivation of R ,

- (i) f Actions as per a homomorphism proceeding $U \Rightarrow$ either $f = 0$ or $U \subset Z(R)$.
- (ii) f actions as per an anti-homomorphism proceeding $U \Rightarrow$ either one $f = 0$ or $U \subset Z(R)$.

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